Warsaw University of Technology
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Institute of Aeronautics and Applied Mechanics

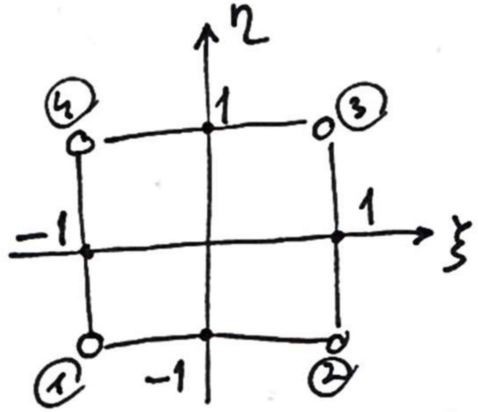
Finite element method (FEM)

4-node quadrilateral element

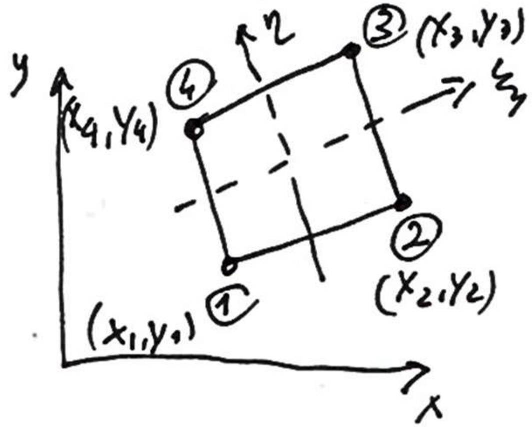
04.2021

4-node quadrilateral element (2D analysis)

natural coordinate system



cartesian coordinate system



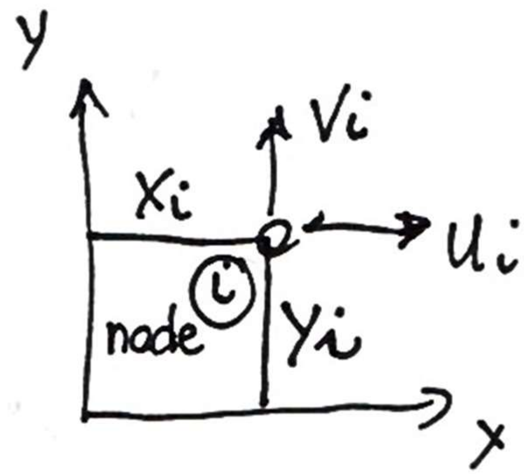
geometry mapping : $(\xi, \eta) \to (x, y)$

$$(-1,-1) \to (x_1, y_1) \quad ; \quad (1,-1) \to (x_2, y_2)$$

$$(1,1) \to (x_3, y_3) \quad ; \quad (-1,1) \to (x_4, y_4)$$

vectors of nodal coordinates

$$\underset{4\times1}{\{X_i\}_e} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_e ; \quad \{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}_e$$



local vector of nodal parameters

$$n = 4, \quad n_p = 2 \rightarrow n_e = n \cdot n_p = 8$$

$$\underset{8\times1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_e$$

isoparametric mapping

$$x = a + b\cdot\xi + c\cdot\eta + d\cdot\xi\eta = \lfloor 1, \xi, \eta, \xi\eta \rfloor \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

(ξ, η) (x, y)

① $(-1,-1) \rightarrow (x_1, y_1)$

$$x_1 = 1\cdot a + (-1)\cdot b + (-1)\cdot c + (-1)\cdot(-1)\cdot d$$

a, b, c, d – constants

② $(1,-1) \rightarrow (x_2, y_2)$

$$x_2 = 1\cdot a + 1\cdot b + (-1)\cdot c + 1\cdot(-1)\cdot d$$

③ $(1,1) \rightarrow (x_3, y_3)$

$$x_3 = 1\cdot a + 1\cdot b + 1\cdot c + 1\cdot 1\cdot d$$

④ $(-1,1) \rightarrow (x_4, y_4)$

$$x_4 = 1\cdot a + (-1)\cdot b + 1\cdot c + (-1)\cdot 1\cdot d$$

$$(\det[A] = -16)$$

$$\{X_i\}_e = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_e = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = [A] \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

$$\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_e = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \cdot \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_e$$

$$X = \lfloor 1, \xi, \eta, \xi\eta \rfloor \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \lfloor 1, \xi, \eta, \xi\eta \rfloor \cdot \underset{4\times 4}{[A]^{-1}} \cdot \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}_e =$$

$$= \lfloor 1, \xi, \eta, \xi\cdot\eta \rfloor \cdot \begin{Bmatrix} \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ -\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ -\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4 \end{Bmatrix} =$$

$$= 1\cdot\left(\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4\right) + \xi\cdot\left(-\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4\right) +$$

$$+ \eta\left(-\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4\right) + \xi\eta\left(\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4\right) =$$

$$= \left(\frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\eta + \frac{1}{4}\xi\eta\right)x_1 + \left(\frac{1}{4} + \frac{1}{4}\xi - \frac{1}{4}\eta - \frac{1}{4}\xi\eta\right)x_2 +$$

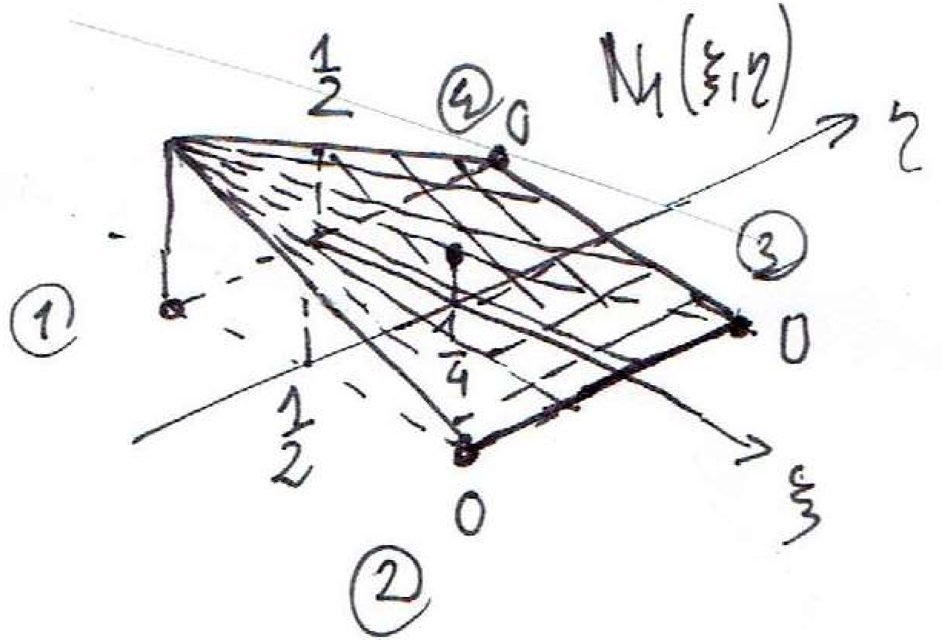
$$+ \left(\frac{1}{4} + \frac{1}{4}\xi + \frac{1}{4}\eta + \frac{1}{4}\xi\eta\right)x_3 + \left(\frac{1}{4} - \frac{1}{4}\xi + \frac{1}{4}\eta - \frac{1}{4}\xi\eta\right)\cdot x_4 =$$

$$= \left(\frac{(1-\xi)(1-\eta)}{4}\right)\cdot x_1 + \frac{(1+\xi)(1-\eta)}{4}x_2 + \frac{(1+\xi)(1+\eta)}{4}x_3 + \frac{(1-\xi)(1+\eta)}{4}x_4$$

$$= N_1(\xi,\eta)\cdot x_1 + N_2(\xi,\eta)\cdot x_2 + N_3(\xi,\eta)x_3 + N_4(\xi,\eta)\cdot x_4$$

shape functions

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

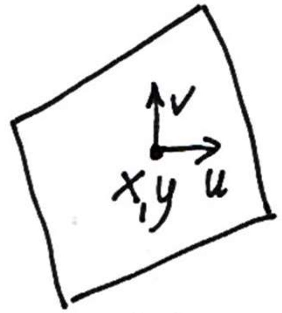


isoparametric mapping

$$x = \underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} \cdot \underset{4\times 1}{\{x_i\}_e}$$

$$y = \underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} \cdot \underset{4\times 1}{\{y_i\}_e}$$

$$\underset{2\times 1}{\{u\}} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underset{2\times 8}{[N(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e}$$



(position and displacement of any point)

where:

$$\underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} = \lfloor N_1, N_2, N_3, N_4 \rfloor$$

$$\underset{2\times 8}{[N(\xi,\eta)]} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

differential operators in the natural coordinate system:

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{[J]} \cdot \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

$[J]$ – Jacobian matrix.

differential operators in the cartesian coordinate system

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1}\cdot\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \frac{1}{\det[J]}\left(\underset{2\times 2}{[J]^{C}}\right)^{T}\cdot\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\det[J]}\cdot\frac{\partial y}{\partial \eta} & -\frac{1}{\det[J]}\cdot\frac{\partial y}{\partial \xi} \\ -\frac{1}{\det[J]}\cdot\frac{\partial x}{\partial \eta} & \frac{1}{\det[J]}\frac{\partial x}{\partial \xi} \end{bmatrix}\cdot\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$\det[J] = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} \Rightarrow$$

$$\frac{\partial x}{\partial \xi} = \frac{\partial\left(\lfloor N(\xi,\eta) \rfloor_{1\times 4} \cdot \{x_i\}_{e\,(4\times 1)}\right)}{\partial \xi} = \frac{\partial \lfloor N(\xi,\eta) \rfloor}{\partial \xi} \cdot \{x_i\}_{e\,(4\times 1)} + \underbrace{\frac{\partial \{x_i\}_e}{\partial \xi}}_{= 0 \text{ (discrete values)}} \cdot \lfloor N(\xi,\eta) \rfloor_{1\times 4} =$$

$$= \left\lfloor \frac{\partial N_1}{\partial \xi}, \frac{\partial N_2}{\partial \xi}, \frac{\partial N_3}{\partial \xi}, \frac{\partial N_4}{\partial \xi} \right\rfloor \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e =$$

$$= \left(-\frac{1}{4}(1-\eta)\right) \cdot x_1 + \frac{1}{4}(1-\eta) \cdot x_2 + \frac{1}{4}(1+\eta) x_3 - \frac{1}{4}(1+\eta) x_4$$

$$\frac{\partial y}{\partial \eta} = \frac{\partial\left(\lfloor N(\xi,\eta) \rfloor_{1\times 4} \cdot \{y_i\}_{e\,(4\times 1)}\right)}{\partial \eta} = \frac{\partial \lfloor N(\xi,\eta) \rfloor_{1\times 4}}{\partial \eta} \cdot \{y_i\}_{e\,(4\times 1)} =$$

$$= \left(-\frac{1}{4}(1-\xi)\right) \cdot y_1 - \frac{1}{4}(1+\xi) \cdot y_2 + \frac{1}{4}(1+\xi) \cdot y_3 + \frac{1}{4}(1-\xi) \cdot y_4$$

$$\frac{\partial y}{\partial \xi} = \frac{\partial\left(\lfloor \underset{1\times 4}{N(\xi,\eta)} \rfloor \cdot \underset{4\times 1}{\{y_i\}}e\right)}{\partial \xi} = \frac{\partial \lfloor \underset{1\times 4}{N(\xi,\eta)} \rfloor}{\partial \xi} \cdot \underset{4\times 1}{\{y_i\}}e =$$

$$= \left(-\frac{1}{4}(1-\eta)\right)\cdot y_1 + \frac{1}{4}(1-\eta)\cdot y_2 + \frac{1}{4}(1+\eta)y_3 - \frac{1}{4}(1+\eta)y_4$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial\left(\lfloor \underset{1\times 4}{N(\xi,\eta)} \rfloor \cdot \underset{4\times 1}{\{x_i\}}e\right)}{\partial \eta} = \frac{\partial \lfloor \underset{1\times 4}{N(\xi,\eta)} \rfloor}{\partial \eta} \cdot \underset{4\times 1}{\{x_i\}}_e =$$

$$= \left(-\frac{1}{4}(1-\xi)\right)x_1 - \frac{1}{4}(1+\xi)\cdot x_2 + \frac{1}{4}(1+\xi)\cdot x_3 + \frac{1}{4}(1-\xi)\cdot x_4$$

$$\frac{\partial}{\partial x} = \frac{1}{\det[J]}\left(\frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial}{\partial y} = \frac{1}{\det[J]}\left(\frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi}\right)$$

gradient matrix for plane stress or plane strain conditions

$$\underset{3\times 2}{[R(x,y)]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} = \underbrace{\frac{1}{\det[J]} \begin{bmatrix} \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} \\ \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} & \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} \end{bmatrix}}_{\underset{3\times 2}{[R(\xi,\eta)]}}$$

strain vector (plane stress, plane strain)

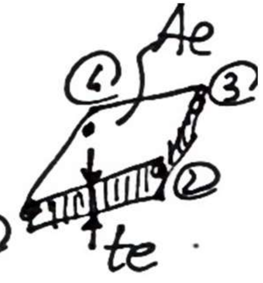
$$\underset{3\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times 2}{[R(\xi,\eta)]}\cdot\underset{2\times 1}{\{u\}} = \underset{3\times 2}{[R(\xi,\eta)]}\cdot\underset{2\times 8}{[N(\xi,\eta)]}\cdot\underset{8\times 1}{\{q\}_e} =$$

$$= \underset{3\times 8}{[B(\xi,\eta)]}\cdot\underset{8\times 1}{\{q\}_e}$$

stress vector (plane stress, plane strain)

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times 3}{[D]}\cdot\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 3}{[D]}\cdot\underset{3\times 8}{[B(\xi,\eta)]}\cdot\underset{8\times 1}{\{q\}_e}$$

elastic strain energy in a finite element



$$U_e = \frac{1}{2} \int_{\Omega_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \underset{3\times 1}{\{\sigma\}} \, d\Omega_e = \frac{1}{2} t_e \int_{A_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \cdot \underset{3\times 1}{\{\sigma\}} \, dA_e =$$

$$= \frac{1}{2} t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor \varepsilon \rfloor \cdot \{\sigma\} \det[J] \, d\xi \, d\eta = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \underset{8\times 8}{[k]_e} \underset{8\times 1}{\{q\}_e}$$

where :

$$[k]_e = \underbrace{t_e \int_{-1}^{1}\int_{-1}^{1} \left(\underset{8\times 3}{[B(\xi,\eta)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(\xi,\eta)]} \det[J(\xi,\eta)] \right) d\xi \, d\eta}_{\text{(numerical integration)}}$$

$$\underset{3\times 8}{\left[B(\xi,\eta)\right]} = \underset{3\times 2}{\left[R(\xi,\eta)\right]}\cdot\underset{2\times 8}{\left[N(\xi,\eta)\right]} = \begin{bmatrix} r_{11} & r_{12}=0 \\ r_{21}=0 & r_{22} \\ r_{31} & r_{32} \\ \| & \| \\ r_{22} & r_{11} \end{bmatrix}\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$r_{11}\cdot N_1 = \frac{\frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta}}{\det[J]}\cdot N_1(\xi,\eta) = \frac{1}{\det[J]}\cdot\left(\frac{\partial y}{\partial \eta}\cdot\frac{\partial N_1}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial N_1}{\partial \eta}\right) =$$

$$= \frac{-\frac{1}{4}(1-\eta)\frac{\partial y}{\partial \eta} + \frac{1}{4}(1-\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{11} = b_{32}$$

$$r_{11}\cdot N_2 = \frac{\frac{1}{4}(1-\eta)\frac{\partial y}{\partial \eta} + \frac{1}{4}(1+\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{13} = b_{34}$$

$$r_{11}\cdot N_3 = \frac{\frac{1}{4}(1+\eta)\frac{\partial y}{\partial \eta} - \frac{1}{4}(1+\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{15} = b_{36}$$

$$r_{11}\cdot N_4 = \frac{-\frac{1}{4}(1+\eta)\frac{\partial y}{\partial \eta} - \frac{1}{4}(1-\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{17} = b_{38}$$

$$r_{22}\cdot N_1 = \frac{\frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi}}{\det[J]}\cdot N_1(\xi,\eta) = \frac{1}{\det[J]}\left(\frac{\partial x}{\partial \xi}\cdot\frac{\partial N_1}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial N_1}{\partial \xi}\right) =$$

$$= \frac{-\frac{1}{4}(1-\xi)\frac{\partial x}{\partial \xi} + \frac{1}{4}(1-\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{22} = b_{31}$$

$$r_{22}\cdot N_2 = \frac{-\frac{1}{4}(1+\xi)\frac{\partial x}{\partial \xi} - \frac{1}{4}(1-\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{24} = b_{33}$$

$$r_{22}\cdot N_3 = \frac{\frac{1}{4}(1+\xi)\frac{\partial x}{\partial \xi} - \frac{1}{4}(1+\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{26} = b_{35}$$

$$r_{22}\cdot N_4 = \frac{\frac{1}{4}(1-\xi)\frac{\partial x}{\partial \xi} + \frac{1}{4}(1+\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{28} = b_{37}$$

$$\underset{3\times 8}{[B]} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} & b_{28} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} & b_{38} \end{bmatrix}$$

$$U_e = \frac{1}{2}\int_{\Omega_e} \lfloor \varepsilon_x, \varepsilon_y, \gamma_{xy} \rfloor \cdot [D] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} d\Omega_e =$$

$$= \underbrace{\frac{1}{2}\int_{\Omega_e} \lfloor \varepsilon_x, \varepsilon_y, 0 \rfloor \underset{3\times 3}{[D]} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \end{Bmatrix} d\Omega_e}_{U_e^{\sigma} \text{ (normal stress)}} + \underbrace{\frac{1}{2}\int_{\Omega_e} \lfloor 0, 0, \gamma_{xy} \rfloor \cdot \underset{3\times 3}{[D]} \cdot \begin{Bmatrix} 0 \\ 0 \\ \gamma_{xy} \end{Bmatrix} d\Omega_e}_{U_e^{\tau} \text{ (shear stress)}}$$

$$\underset{3\times 1}{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \end{Bmatrix}} = \underset{3\times 2}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \end{bmatrix}} \underset{2\times 1}{\{u\}} = \underset{3\times 2}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \end{bmatrix}} \cdot \underset{2\times 8}{[N(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e} = \underset{3\times 8}{[B_\varepsilon]} \cdot \underset{8\times 1}{\{q\}_e}$$

$$\underset{3\times 1}{\begin{Bmatrix} 0 \\ 0 \\ \gamma_{xy} \end{Bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \cdot \underset{2\times 1}{\{u\}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \underset{2\times 8}{[N(\xi,\eta)]} \underset{8\times 1}{\{q\}_e} = \underset{3\times 8}{[B_\gamma]} \cdot \underset{8\times 1}{\{q\}_e}$$

$$[B_\varepsilon] = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{18} \\ b_{21} & b_{22} & \cdots & b_{28} \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad [B_\gamma] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & \cdots & b_{38} \end{bmatrix}$$

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[K_{\varepsilon}]_e} \underset{8\times 1}{\{q\}_e}$$

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_{\gamma}]_e} \underset{8\times 1}{\{q\}_e}$$

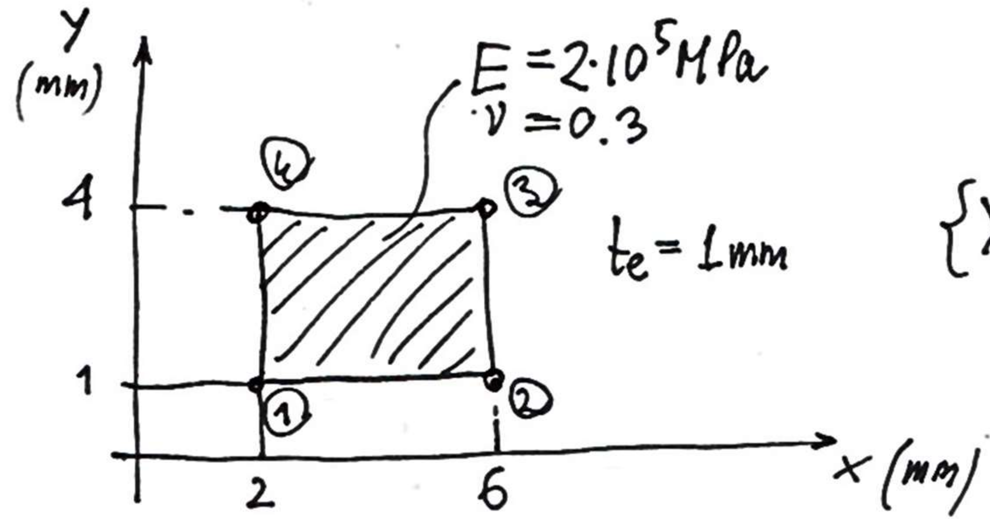
where:

$$[K_{\varepsilon}]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_{\varepsilon}]^T [D] [B_{\varepsilon}] \cdot \det [J(\xi,\eta)] \right) d\xi \, d\eta$$

$$[K_{\gamma}]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_{\gamma}]^T [D] [B_{\gamma}] \cdot \det [J(\xi,\eta)] \right) d\xi \, d\eta$$

$$[K]_e = [K_{\varepsilon}]_e + [K_{\gamma}]_e$$

Example : 4-node quadrilateral FE.



$$\{X_i\}_e = \begin{Bmatrix} 2 \\ 6 \\ 6 \\ 2 \end{Bmatrix} \; ; \; \{y_i\}_e = \begin{Bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{Bmatrix}$$

$$x_4 = x_1 \, , \; x_3 = x_2 \, , \; y_2 = y_1 \; ; \; y_4 = y_3$$

$$\frac{\partial x}{\partial \xi} = \left(-\frac{1}{4}(1-\eta)\right)\cdot x_1 + \frac{1}{4}(1-\eta)x_2 + \frac{1}{4}(1+\eta)x_3 - \frac{1}{4}(1+\eta)x_4 =$$

$$= \left(-\frac{1}{4}(1-\eta) - \frac{1}{4}(1+\eta)\right)\cdot x_1 + \left(\frac{1}{4}(1-\eta) + \frac{1}{4}(1+\eta)\right)x_2 =$$

$$= -\frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(6-2) = 2\,mm$$

$$\frac{\partial y}{\partial \eta} = \left(-\frac{1}{4}(1-\xi)\right)\cdot y_1 - \frac{1}{4}(1+\xi)\cdot y_2 + \frac{1}{4}(1+\xi)\cdot y_3 + \frac{1}{4}(1-\xi)\cdot y_4 =$$

$$= \left(-\frac{1}{4}(1-\xi) - \frac{1}{4}(1+\xi)\right)y_1 + \left(\frac{1}{4}(1+\xi) + \frac{1}{4}(1-\xi)\right)y_3 =$$

$$= -\frac{1}{2}y_1 + \frac{1}{2}y_3 = \frac{1}{2}(y_3 - y_1) = \frac{1}{2}(4-1) = 1{,}5\,mm$$

$$\frac{\partial y}{\partial \xi} = \left(-\frac{1}{4}(1-\eta)\right)\cdot y_1 + \frac{1}{4}(1-\eta)\cdot \overset{(y_1)}{\overset{\|}{y_2}} + \frac{1}{4}(1+\eta)\cdot y_3 - \frac{1}{4}(1+\eta)\cdot \overset{(y_3)}{\overset{\|}{y_4}} =$$

$$= 0\cdot y_1 + 0\cdot y_3 = 0$$

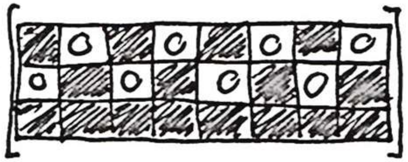
$$\frac{\partial x}{\partial \eta} = \left(-\frac{1}{4}(1-\xi)\right)\cdot x_1 - \frac{1}{4}(1+\xi)\cdot x_2 + \frac{1}{4}(1+\xi)\cdot \overset{(x_2)}{\overset{\|}{x_3}} + \frac{1}{4}(1-\xi)\cdot \overset{(x_1)}{\overset{\|}{x_4}} =$$

$$= 0\cdot x_1 + 0\cdot x_2 = 0$$

$$\det[J] = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial x}{\partial \eta} = 2\,mm\cdot 1.5\,mm - 0\cdot 0 = 3\,mm^2$$

Strain-displacement matrix:

$$\underset{3\times 8}{[B]} =$$



$$b_{12} = b_{14} = b_{16} = b_{18} = b_{21} = b_{23} = b_{25} = b_{27} = 0$$

$$b_{11} = b_{32} = \frac{-\frac{1}{4}(1-\eta)\cdot 1.5mm + \frac{1}{4}(1-\xi)\cdot 0mm}{3mm^2} = -\frac{1}{8}(1-\eta)\,\frac{1}{mm}$$

$$b_{13} = b_{34} = \frac{\frac{1}{4}(1-\eta)\cdot 1.5mm + \frac{1}{4}(1+\xi)\cdot 0mm}{3mm^2} = \frac{1}{8}(1-\eta)\,\frac{1}{mm}$$

$$b_{15} = b_{36} = \frac{\frac{1}{4}(1+\eta)\cdot 1.5mm - \frac{1}{4}(1+\xi)\cdot 0mm}{3mm^2} = \frac{1}{8}(1+\eta)\,\frac{1}{mm}$$

$$b_{17} = b_{38} = \frac{-\frac{1}{4}(1+\eta)\cdot 1.5mm - \frac{1}{4}(1-\xi)\cdot 0mm}{3mm^2} = -\frac{1}{8}(1+\eta)\,\frac{1}{mm}$$

Strain-displacement matrix:

$$b_{22} = b_{31} = \frac{-\frac{1}{4}(1-\xi)\cdot 2mm + \frac{1}{4}(1-\eta)\cdot 0mm}{3mm^2} = -\frac{1}{6}(1-\xi)\ \frac{1}{mm}$$

$$b_{24} = b_{33} = \frac{-\frac{1}{4}(1+\xi)\cdot 2mm - \frac{1}{4}(1-\eta)\cdot 0mm}{3mm^2} = -\frac{1}{6}(1+\xi)\ \frac{1}{mm}$$

$$b_{26} = b_{35} = \frac{\frac{1}{4}(1+\xi)\cdot 2mm - \frac{1}{4}(1+\eta)\cdot 0mm}{3mm^2} = \frac{1}{6}(1+\xi)\ \frac{1}{mm}$$

$$b_{28} = b_{37} = \frac{\frac{1}{4}(1-\xi)\cdot 2mm + \frac{1}{4}(1+\eta)\cdot 0mm}{3mm^2} = \frac{1}{6}(1-\xi)\ \frac{1}{mm}$$

$$\underset{\substack{3\times 8 \\ \frac{1}{mm}}}{[B(\xi,\eta)]} = \begin{bmatrix} -(1-\eta)/8 & 0 & (1-\eta)/8 & 0 & (1+\eta)/8 & 0 & -(1+\eta)/8 & 0 \\ 0 & -(1-\xi)/6 & 0 & -(1+\xi)/6 & 0 & (1+\xi)/6 & 0 & (1-\xi)/6 \\ -(1-\xi)/6 & -(1-\eta)/8 & -(1+\xi)/6 & (1-\eta)/8 & (1+\xi)/6 & (1+\eta)/8 & (1-\xi)/6 & -(1+\eta)/8 \end{bmatrix}$$

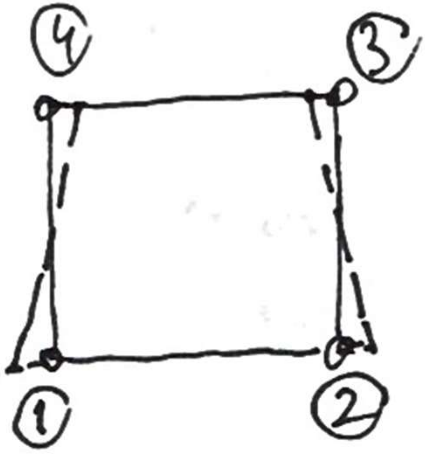
Strain-displacement matrix:

$$\underset{3\times 8}{[B(\xi,\eta)]} = \underset{3\times 8}{[B_\varepsilon(\xi,\eta)]} + \underset{3\times 8}{[B_\gamma(\xi,\eta)]}$$

$$\underset{3\times 8}{[B_\varepsilon(\xi,\eta)]} = \begin{bmatrix} -(1-\eta)/8 & 0 & (1-\eta)/8 & 0 & (1+\eta)/8 & 0 & -(1+\eta)/8 & 0 \\ 0 & -(1-\xi)/6 & 0 & -(1+\xi)/6 & 0 & (1+\xi)/6 & 0 & (1-\xi)/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{mm}$$

$$\underset{3\times 8}{[B_\gamma(\xi,\eta)]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(1-\xi)/6 & -(1-\eta)/8 & -(1+\xi)/6 & (1-\eta)/8 & (1+\xi)/6 & (1+\eta)/8 & (1-\xi)/6 & -(1+\eta)/8 \end{bmatrix} \frac{1}{mm}$$

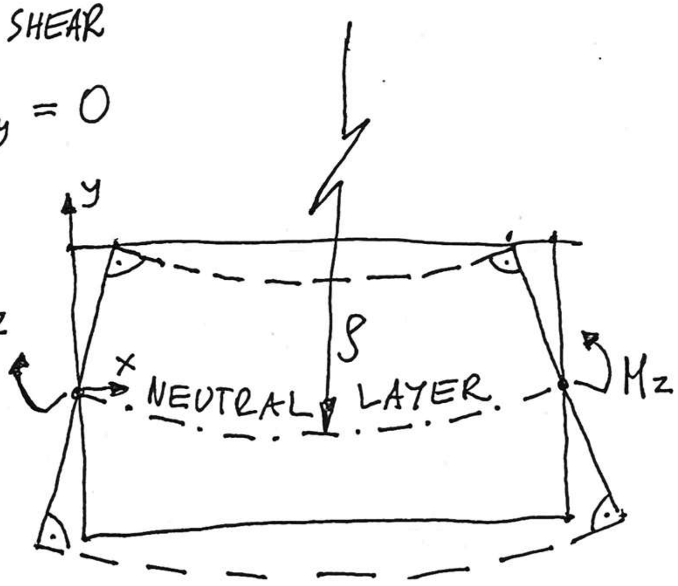
Case 1. „Bending"



$$\underset{\substack{8\times1 \\ (mm)}}{\{q\}_e} = \begin{array}{c} \left\{\begin{array}{c} -0.001 \\ 0 \\ 0.001 \\ 0 \\ -0.001 \\ 0 \\ 0.001 \\ 0 \end{array}\right\}_e \end{array} \begin{array}{l} (u_1) \\ \\ (u_2) \\ \\ (u_3) \\ \\ (u_4) \\ \\ \end{array}$$

PURE BENDING – NO SHEAR

$$\tau_{xy} = 0 \Rightarrow \gamma_{xy} = 0$$



$$\boxed{\varepsilon_x = -\frac{1}{\rho}\cdot y}$$

$$y = \frac{3mm}{2} \Rightarrow \varepsilon_x = -0.5\cdot 10^{-3} \qquad \rho = \frac{1.5}{0.5}\cdot 10^3 mm = 3000 mm$$

$$\boxed{\sigma_x = E\varepsilon_x} = -\frac{E}{\rho}\cdot y = -\frac{2\cdot 10^5 MPa}{3000\ mm}\cdot y = -\frac{200}{3}\frac{MPa}{mm}\cdot y$$

$$\sigma_x(1.5mm) = -\frac{200}{3}\cdot 1.5\ MPa = -100\ MPa$$

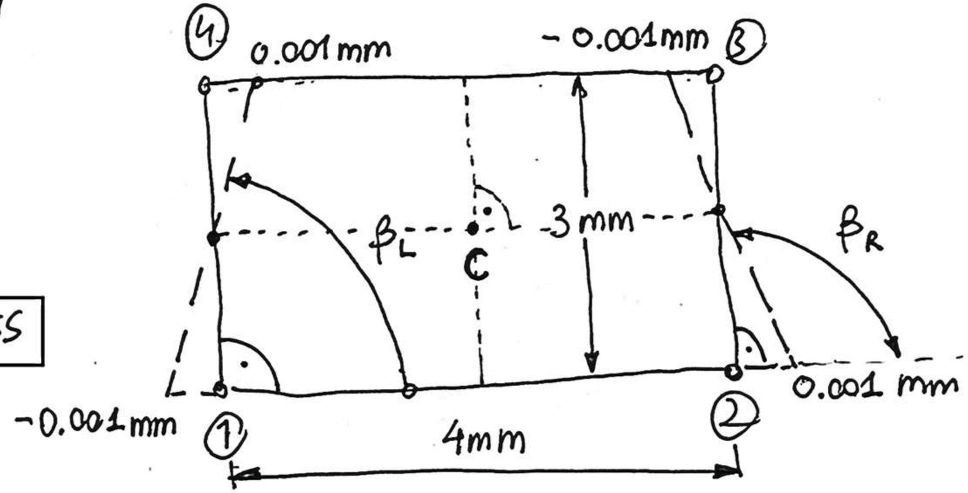
$$\boxed{\sigma_x = -\frac{M_z\cdot y}{J_z}} \Rightarrow -\frac{M_z}{J_z} = -\frac{E}{\rho} \Rightarrow \boxed{\frac{1}{\rho} = \frac{M_z}{EJ_z}}$$

Case 1. „Bending"

$E = 2\cdot 10^5$ MPa

$\nu = 0.3$

PLANE STRESS



Strain components:

$$\varepsilon_x^{③} = \varepsilon_x^{④} = \frac{\Delta l_{34}}{l_{34}} = \frac{-0.002\,\text{mm}}{4\ \text{mm}} = -0.5\cdot 10^{-3}$$

$$\varepsilon_x^{①} = \varepsilon_x^{②} = \frac{\Delta l_{12}}{l_{12}} = \frac{0.002\ \text{mm}}{4\,\text{mm}} = 0.5\cdot 10^{-3}$$

$$\varepsilon_y^{①} = \varepsilon_y^{②} = \varepsilon_y^{③} = \varepsilon_y^{④} = 0$$

$$\gamma_{xy}^{①} = \gamma_{xy}^{④} = \frac{\pi}{2} - \beta_L \cong \frac{(0.001-(-0.001))\,\text{mm}}{3\,\text{mm}} = 0.667\cdot 10^{-3}$$

$$\gamma_{xy}^{③} = \gamma_{xy}^{②} = \frac{\pi}{2} - \beta_R \cong \frac{(0.001-0.001)\,\text{mm}}{3\,\text{mm}} = -0.667\cdot 10^{-3}$$

Stress components:

$$\sigma_x^{①} = \sigma_x^{②} = \frac{E}{1-\nu^2}\left(\varepsilon_x^{①} + \nu\,\varepsilon_y^{①}\right) = \frac{2\cdot 10^5\ \mathrm{MPa}}{1-0.3^2}\cdot 0.5\cdot 10^{-3} = 109.89\ \mathrm{MPa}$$

$$\sigma_x^{③} = \sigma_x^{④} = \frac{E}{1-\nu^2}\left(\varepsilon_x^{③} + \nu\,\varepsilon_y^{③}\right) = -109.89\ \mathrm{MPa}$$

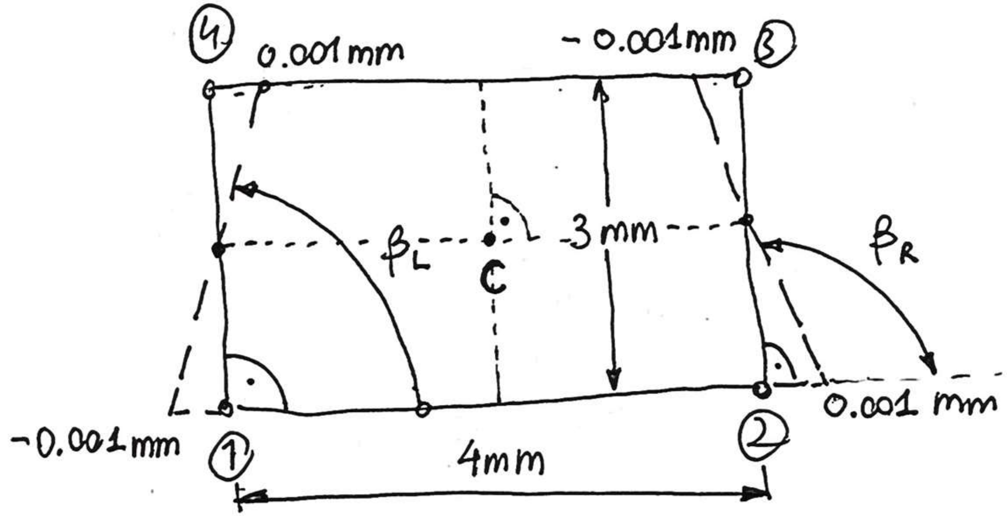
$$\sigma_y^{①} = \sigma_y^{②} = \frac{E}{1-\nu^2}\left(\varepsilon_y^{①} + \nu\,\varepsilon_x^{①}\right) = \frac{2\cdot 10^5\ \mathrm{MPa}}{1-0.3^2}\cdot 0.3\cdot 0.5\cdot 10^{-3} = 32.97\ \mathrm{MPa}$$

$$\sigma_y^{③} = \sigma_y^{④} = \frac{E}{1-\nu^2}\left(\varepsilon_y^{③} + \nu\,\varepsilon_y^{③}\right) = -32.97\ \mathrm{MPa}$$

$$\tau_{xy}^{①} = \tau_{xy}^{④} = \gamma_{xy}^{①}\cdot G = 0.667\cdot 10^{-3}\cdot\frac{2\cdot 10^5\ \mathrm{MPa}}{2(1+0.3)} = 51.28\ \mathrm{MPa}$$

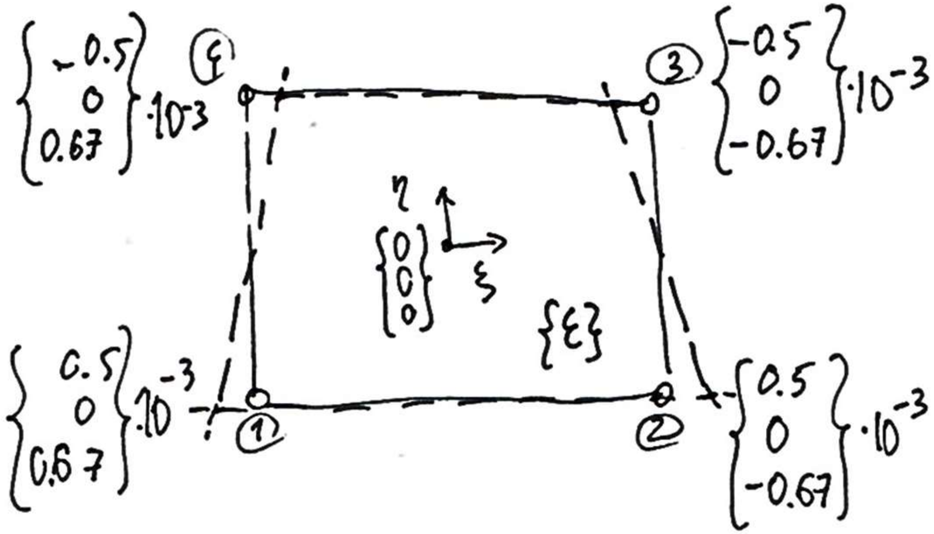
$$\tau_{xy}^{②} = \tau_{xy}^{③} = \gamma_{xy}^{②}\cdot G = -51.28\ \mathrm{MPa}$$

Strain and stress components at the center (point C):

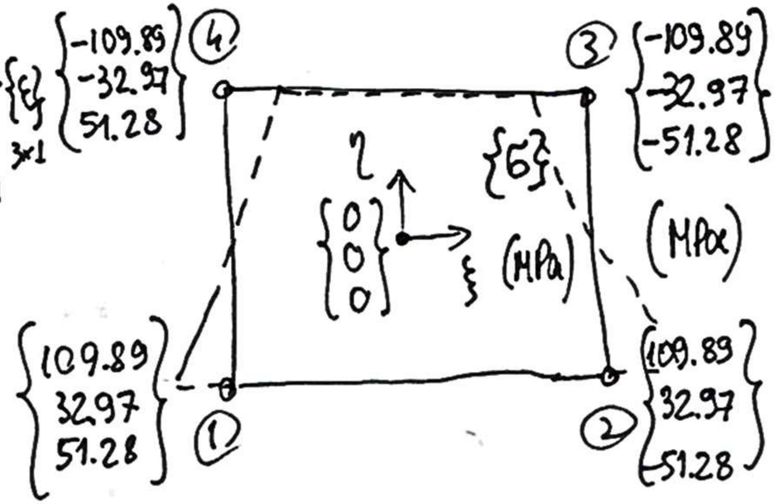


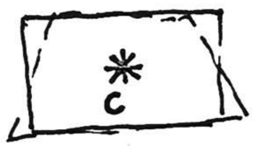
$$\varepsilon_x^c = 0 \ , \quad \varepsilon_y^c = 0 \ , \quad \gamma_{xy}^c = 0 \quad \Rightarrow \quad \begin{aligned} \sigma_x^c &= 0 \\ \sigma_y^c &= 0 \\ \tau_{xy}^c &= 0 \end{aligned}$$

$$\underset{3\times1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times8}{[B(\xi,\eta)]} \cdot \underset{8\times1}{\{q\}_e}$$



$$\underset{3\times1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times3}{[D]} \cdot \underset{3\times1}{\{\varepsilon\}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \cdot \underset{3\times1}{\{\varepsilon\}}$$



NUMERICAL INTEGRATION , $n = 1$  $w_1 \cdot w_1 = 4$

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \underset{8\times 8}{[k]_e} \cdot \underset{8\times 1}{\{q\}_e} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \int_{\Omega_e} \underset{8\times 3}{[B]^T} \underset{3\times 3}{[D]} \underset{3\times 8}{[B]} \, d\Omega_e \cdot \underset{8\times 1}{\{q\}_e} =$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B(\xi,\eta)]^T} \underset{3\times 3}{[D]} \underset{3\times 8}{[B(\xi,\eta)]} \det[J(\xi,\eta)] \, d\xi \, d\eta \cdot \underset{8\times 1}{\{q\}_e} = \left| \begin{matrix} n = 1 \\ \xi_1 = 0 \\ \eta_1 = 0 \end{matrix} \right| =$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \, t_e \cdot \underset{8\times 3}{[B(0,0)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(0,0)]} \cdot \det \underset{\Uparrow}{[J(0,0)]} \cdot w_1 \cdot w_1 \cdot \underset{8\times 1}{\{q\}_e} = 0$$

$$\frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} = 2\,mm \cdot 1.5\,mm - 0 \cdot 0 = 3\,mm^2$$

$$\underset{3\times 3}{[D]} = \frac{2 \cdot 10^5}{1 - 0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-0.3) \end{bmatrix} MPa$$

$$\underset{3\times 8}{[B(0,0)]} = \begin{bmatrix} -1/8 & 0 & 1/8 & 0 & 1/8 & 0 & -1/8 & 0 \\ 0 & -1/6 & 0 & -1/6 & 0 & 1/6 & 0 & 1/6 \\ -1/6 & -1/8 & -1/6 & 1/8 & 1/6 & 1/8 & 1/6 & -1/8 \end{bmatrix} \frac{1}{mm}$$

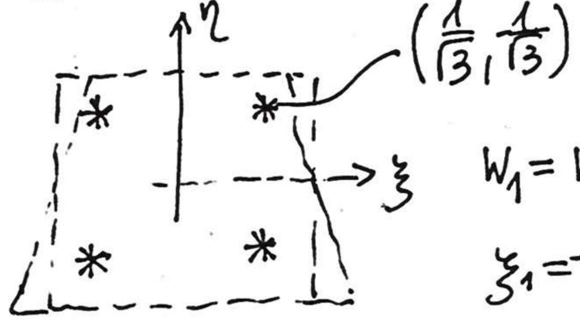
$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times3}{\lfloor \varepsilon \rfloor} \cdot \underset{3\times1}{\{\sigma\}} d\Omega_e = \frac{1}{2} t_e \int_{-1}^{1}\int_{-1}^{1} \underset{1\times3}{\lfloor \varepsilon(\xi,\eta) \rfloor} \cdot \underset{3\times1}{\{\sigma(\xi,\eta)\}} \cdot \det[J(\xi,\eta)] \cdot d\xi d\eta =$$

$$= \left| \begin{array}{l} n = 1 \\ \xi_1 = 0 \\ \eta_1 = 0 \\ W_1 W_1 = 4 \end{array} \right| = \frac{1}{2} t_e \underset{1\times3}{\lfloor \varepsilon(0,0) \rfloor} \cdot \underset{3\times1}{\{\sigma(0,0)\}} \cdot \det[J(0,0)] \cdot W_1 W_1 = 0$$

$$\lfloor \varepsilon(0,0) \rfloor \Leftarrow \lfloor \underset{0}{\underset{\parallel}{\varepsilon_x^c}}, \underset{0}{\underset{\parallel}{\varepsilon_y^c}}, \underset{0}{\underset{\parallel}{\gamma_{xy}^c}} \rfloor \qquad \{\sigma(0,0)\} \Leftarrow \begin{Bmatrix} \sigma_x^c \\ \sigma_y^c \\ \tau_{xy}^c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow U_e^{\tau} = 0 \, , \; U_e^{\sigma} = 0$$

NUMERICAL INTEGRATION, $n=2$



$W_1 = W_2 = 1$

$\xi_1 = -\frac{1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{3}}$

$\eta_1 = -\frac{1}{\sqrt{3}}, \quad \eta_2 = \frac{1}{\sqrt{3}}$

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{\left[B(\xi,\eta)\right]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{\left[B(\xi,\eta)\right]} \det\left[J(\xi,\eta)\right] d\xi\, d\eta \cdot \underset{8\times 1}{\{q\}_e} =$$

$$\underset{8\times 8}{\left[f(\xi,\eta)\right]} = \underset{8\times 3}{\left[B(\xi,\eta)\right]^T} \cdot \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{\left[B(\xi,\eta)\right]} \cdot \det\left[J(\xi,\eta)\right]$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 8}{\left[f(\xi,\eta)\right]} d\xi\, d\eta \cdot \underset{8\times 1}{\{q\}_e} =$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} t_e \cdot \left(\underset{8\times 8}{\left[f(\xi_1,\eta_1)\right]} \cdot W_1 W_1 + \underset{8\times 8}{\left[f(\xi_2,\eta_1)\right]} \cdot W_2 W_1 + \underset{8\times 8}{\left[f(\xi_2,\eta_2)\right]} \cdot W_2 W_2 + \underset{8\times 8}{\left[f(\xi_1,\eta_2)\right]} \cdot W_1 W_2 \right) \cdot \underset{8\times 1}{\{q\}_e} = 0.1783 \text{ Nmm}$$

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \cdot \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B_{\varepsilon}(\xi,\eta)]^T} \cdot \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B_{\varepsilon}(\xi,\eta)]} \det[J(\xi,\eta)]\, d\xi\, d\eta\, \{q\}_e =$$

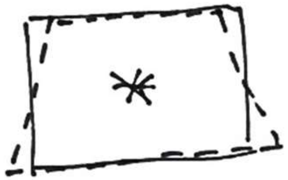
$$= 0.1099 \text{ Nmm}$$

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \cdot \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B_{\gamma}(\xi,\eta)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B_{\gamma}(\xi,\eta)]} \det[J(\xi,\eta)]\, d\xi\, d\eta \cdot \{q\}_e =$$

$$= 0.0684 \text{ Nmm}$$

elastic strain energy
(Gauss method)

$n = 1$



$W_1 W_1 = 4$

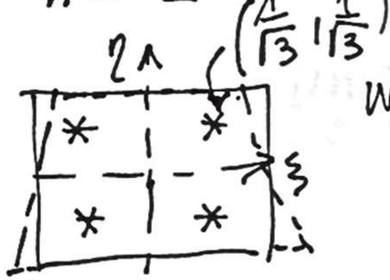
$$U_e = \tfrac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k]_e} \cdot \underset{8\times 1}{\{q\}_e} = 0$$

$$U_e^{\sigma} = \tfrac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_\varepsilon]_e} \underset{8\times 1}{\{q\}_e} = 0$$

$$U_e^{\tau} = \tfrac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_\gamma]} \{q\}_e = 0$$

(zero energy mode = hourglassing)

$n = 2$



$W_1 = W_2 = 1$

$$U_e = 0.1783 \quad Nmm$$

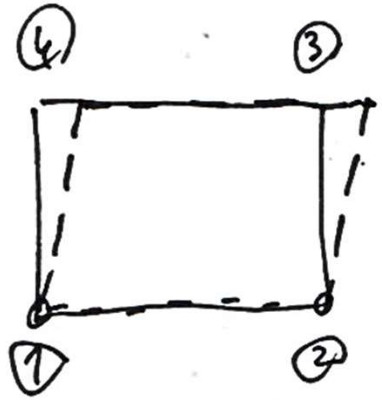
$$U_e^{\sigma} = 0.1099 \quad Nmm$$

$$U_e^{\tau} = 0.0684 \quad Nmm$$

$(U_e^{\tau} = 38\,\% \cdot U_e \; ?)$

(shear locking)

Case 2. „Shear"



$$\{q\}_e = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.001 \\ 0 \\ 0.001 \\ 0 \end{Bmatrix} \begin{array}{l} \\ \\ \\ \\ (u_3) \\ \\ (u_4) \\ \\ \end{array}$$

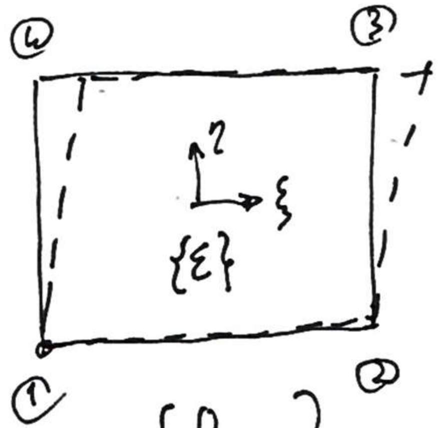
8×1

(mm)

$$\gamma_{xy} \cong \frac{0.001\,mm}{l_{23}} = \frac{0.001\,mm}{3\ mm} = 0.33\cdot10^{-3}$$

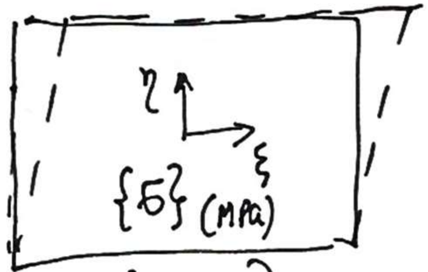
$$\tau_{xy} = G\cdot\gamma_{xy} = \frac{E}{2(1+\nu)}\cdot\gamma_{xy} = 25.64\,MPa$$

$$\underset{3\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times 8}{[B(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e}$$



$$\begin{Bmatrix} 0 \\ 0 \\ 0.33\cdot 10^{-3} \end{Bmatrix} \text{ – uniform}$$

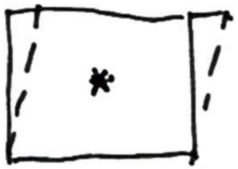
$$\underset{3\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times 3}{[D]} \cdot \underset{3\times 1}{\{\varepsilon\}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$



$$\begin{Bmatrix} 0 \\ 0 \\ 25.641 \end{Bmatrix} \text{ – uniform}$$

elastic strain energy

$n = 1$



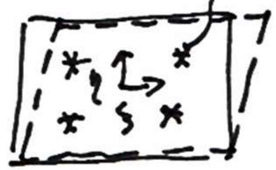
$w_1 w_4 = 4$

$$U_e = \frac{1}{2} \lfloor q \rfloor_e \cdot [k]_e \cdot \{q\}_e = 0.0513 \text{ Nmm}$$

$U_e^{\sigma} = 0$, $U_e^{\tau} = U_e$

$n = 2$

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

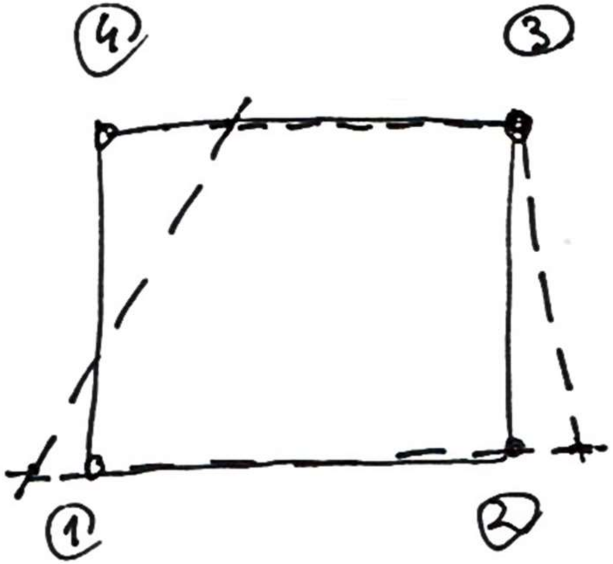


$w_1 = w_2 = 1$

$U_e = 0.0513 \text{ Nmm}$

$U_e^{\sigma} = 0$, $U_e^{\tau} = U_e$

Case 3 = Case 1 + Case 2 „bending + shear"

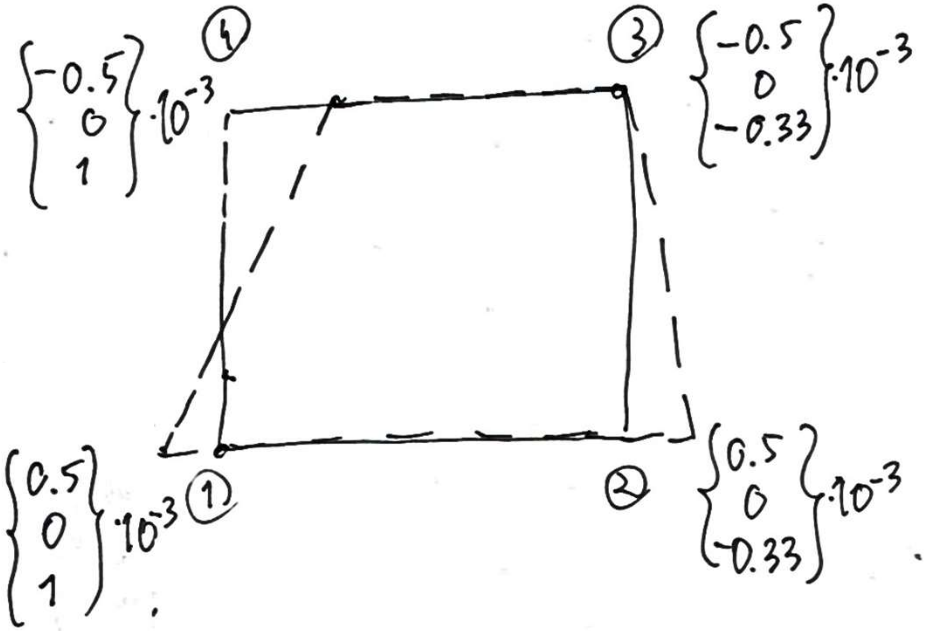


$$\{q\}_{e} = \begin{Bmatrix} -0.001 \\ 0 \\ 0.001 \\ 0 \\ 0 \\ 0 \\ 0.002 \\ 0 \end{Bmatrix} \begin{array}{l} (u_1) \\ \\ (u_2) \\ \\ \\ \\ (u_4) \\ \\ \end{array}$$

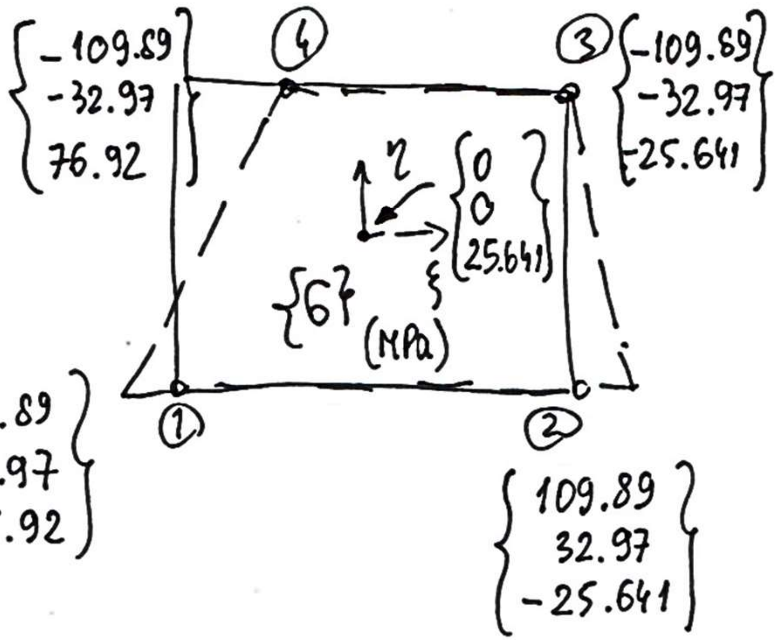
8×1

(mm)

$$\underset{3\times1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times8}{\left[B(\xi,\eta)\right]} \cdot \underset{8\times1}{\{q\}_e}$$



$$\underset{3\times1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times3}{[D]} \cdot \underset{3\times1}{\{\varepsilon\}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$



elastic strain energy

n = 1

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k]_e} \underset{8\times 1}{\{q\}_e} = 0.0513 \text{ Nmm}$$

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_\varepsilon]} \{q\}_e = 0 \text{ Nmm}$$

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} [k_\gamma] \{q\}_e = 0.0513 \text{ Nmm} = U_e$$

n = 2

$$U_e = 0.22955 \text{ Nmm}$$

$$U_e^{\sigma} = 0.1099 \text{ Nmm}$$

$$U_e^{\tau} = 0.1197 \text{ Nmm}$$

$$U_e^{\tau} = 52\%\, U_e$$

$$U_e^{\tau}(\text{case3}) = U_e^{\tau}(\text{case1}) + U_e^{\tau}(\text{case2}) =$$

$$= (0.0513 + 0.0684) \text{ Nmm} =$$

$$= 0.1197 \text{ Nmm}$$

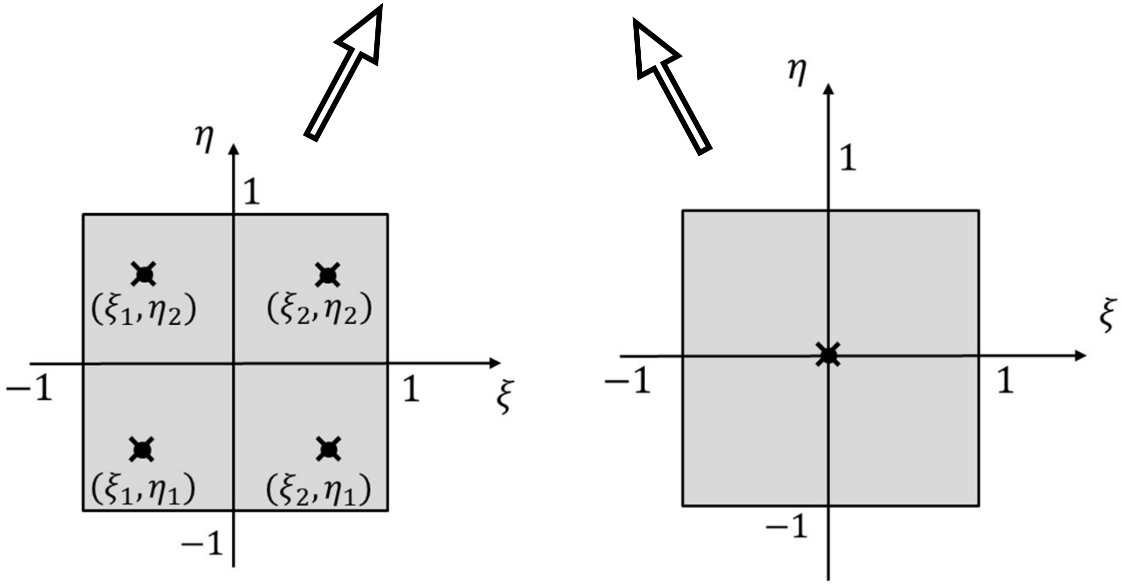
shear locking

Summary

CASE / U_e [Nmm]	n=1			n=2		
	U_e^{σ}	U_e^{τ}	U_e	U_e^{σ}	U_e^{τ}	U_e
1. „BENDING"	0	0	0	0.1099	0.0684	0.1783
2. „SHEAR"	0	0.0513	0.0513	0	0.0513	0.0513
3. „BENDING + SHEAR"	0 = (0 + 0)	0.0513 = (0 + 0.0513)	0.0513 = (0 + 0.0513)	0.1099 = (0 + 0.1099)	0.1197 = (0.0513 + 0.0684)	0.22955 = 0.0513 + 0.1783

Conclusion: $[k]_e = [k_\varepsilon]_e + [k_\gamma]_e$

(element technology)



$$[k_\varepsilon]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_\varepsilon]^T [D] [B_\varepsilon] \cdot \det[J(\xi,\eta)] \right) d\xi \, d\eta$$

$$[k_\gamma]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_\gamma]^T [D] [B_\gamma] \det[J(\xi,\eta)] \right) d\xi \, d\eta$$

Mixed quadrature rule

Full integration ($n = 2$) :

$$U_e^{\sigma} = \frac{1}{2} \underbrace{\lfloor q \rfloor_e}_{1\times 8} \underbrace{[k_\varepsilon]}_{8\times 8} \{q\}_e = 0.1099 \ Nmm$$

Reduced integration ($n = 1$) :

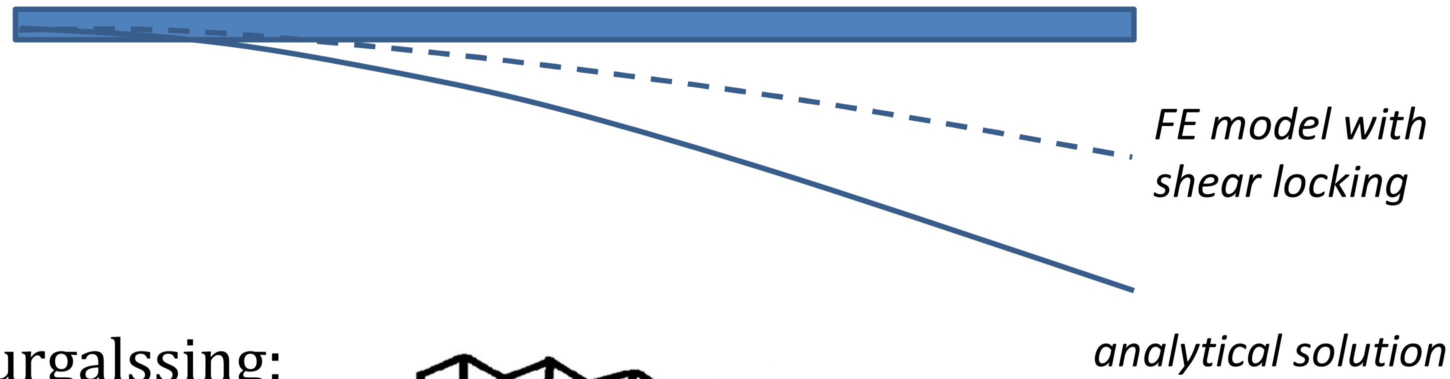
$$U_e^{\tau} = \frac{1}{2} \underbrace{\lfloor q \rfloor_e}_{1\times 8} [k_\gamma] \{q\}_e = 0.0513 \ Nmm$$

$$= U_e$$

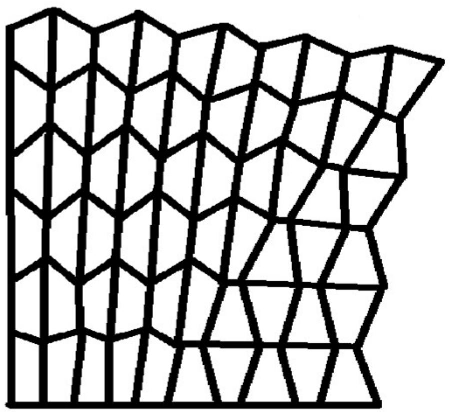
$$U_e = U_e^{\sigma} + U_e^{\tau} = 0.16117 \ Nmm$$

$$U_e \,(case 3) = U_e^{\sigma}(case 1) + U_e^{\tau}(case 2)$$

– shear locking:



– hourgalssing:



– volumetric locking in nearly incompressible materials ($\nu \cong 0.5$)

Element technology - linear materials

Element	Stress State	Poisson's ratio <= 0.49	Poisson's ratio > 0.49 (or anisotropic materials)
PLANE182	Plane stress	KEYOPT(1) = 2 (Enhanced strain formulation)	KEYOPT(1) = 2 (Enhanced strain formulation)
	Not plane stress	KEYOPT(1) = 3 (Simplified enhanced strain formulation)	KEYOPT(1) = 2 (Enhanced strain formulation)
PLANE183	Plane stress	No change	No change
	Not plane stress	No change	No change
SOLID185		KEYOPT(2) = 3 (Simplified enhanced strain formulation)	KEYOPT(2) = 2 (Enhanced strain formulation)
SOLID186		KEYOPT(2) = 0 (Uniform reduced integration)	KEYOPT(2) = 0 (Uniform reduced integration)
SHELL281		No change	No change

(+extra displacement shapes functions)

Shear Locking and Hourglassing in MSC Nastran, ABAQUS, and ANSYS

Eric Qiuli Sun

Abstract

A solid beam and a composite beam were used to compare how MSC Nastran, ABAQUS, and ANSYS handled the numerical difficulties of shear locking and hourglassing. Their tip displacements and first modes were computed, normalized, and listed in multiple tables under various situations. It was found that fully integrated first order solid elements in these three finite element codes exhibited similar shear locking. It is thus recommended that one should avoid using this type of element in bending applications and modal analysis. There was, however, no such shear locking with fully integrated second order solid elements. Reduced integration first order solid elements in ABAQUS and ANSYS suffered from hourglassing when a mesh was coarse. If there was only one layer of elements, the reported first mode of the beam examples from ABAQUS and ANSYS was excessively smaller than the converged solutions due to hourglassing. At least four layers of elements should, therefore, be used in ABAQUS and ANSYS. MSC Nastran outperformed ABAQUS and ANSYS by virtually eliminating the annoying hourglassing of reduced integration first order 3D solid elements because it employed bubble functions to control the propagation of non-physical zero-energy modes. Even if there was only one layer of such elements, MSC Nastran could still manage to produce reasonably accurate results. This is very convenient because it is much less prone to errors when using reduced integration first order 3D solid elements in MSC Nastran.

https://moodle.umontpellier.fr/pluginfile.php/480056/mod_resource/content/0/Sun-ShearLocking-Hourglassing.pdf